

Open problems concerning the Hölder continuity of the direction of vorticity for the Navier-Stokes equations.

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Abstract

In these notes we consider solutions u to the evolution three dimensional Navier-Stokes equations in the whole space. We set $\omega = \nabla \times u$, the vorticity of u . For brevity, we set $L^s(L^r) = L^s(0, T; L^r(\mathbb{R}^3))$, and similar. Our study concerns mainly the relation between β -Hölder continuity assumptions on the direction of the vorticity and induced regularity results like $\omega \in L^\infty(L^r)$. It is well known that for $\beta = \frac{1}{2}$ one gets $\omega \in L^\infty(L^2)$. In particular, $u \in L^\infty(L^6)$ follows. This shows that $\frac{1}{2}$ -Hölder continuity implies strong regularity for the velocity u , by a classical result. On the other hand, it looks quite predictable that a strictly decreasing perturbation of β near $\frac{1}{2}$, should induce a strictly decreasing perturbation for r near 2. This possibility would imply strong regularity for values $\beta < \frac{1}{2}$, since strong regularity holds if merely $\omega \in L^\infty(L^r)$, for some $r \geq \frac{3}{2}$. This result would be in contrast with a previous conjecture which suggests that $\beta = \frac{1}{2}$ is the smallest value enjoying (in some non rigorous sense, see below) the above strong regularization property. In fact, in the sequel, the smallest value β for which we will be able to prove that $\omega \in L^s(L^r)$ for any $r \in (1, 2]$, is still $\beta = \frac{1}{2}$. This conclusion reenforces the above conjecture on the particular significance of the value $\beta = \frac{1}{2}$. But, on the other hand, it also shows the lack of a complete understanding of the full phenomena since the above perturbation argument looks well-founded. In the last section we discuss some related open problems, consistent with our calculations. We hope that the approach bellow should help readers interesting in carrying on this investigation.

Key words: Evolution Navier-Stokes equations; Direction of Vorticity effects; Regularity of solutions.

1 Introduction. Motivation and conclusions.

In the following we consider the evolution Navier-Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla) u - \Delta u + \nabla p = 0, \\ \nabla \cdot u = 0 \quad \text{in } \mathbb{R}^3 \times (0, T], \\ u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

where the divergence free initial data u_0 belongs (for instance) to the Sobolev space $H^1(\mathbb{R}^3)$. For simplicity, we assume that external force vanishes, and viscosity is equal to one. If we want solutions which are regular including the time $t = 0$, then u_0 should be assumed more regular. This kind of problem is out of real interest here. We consider the simplest situation, the whole space case, even if no-slip boundary conditions could be considered.

We will not repeat well know notation as, for instance, Sobolev spaces notation, and so on. For brevity, we set $L^s(L^r) = L^s(0, T; L^r(\mathbb{R}^3))$, and similar. Solutions $u \in L^2(0, T; H^1(\mathbb{R}^3)) \cap L^\infty(0, T; L^2(\mathbb{R}^3))$ are defined in the well known Leray-Hopf weak sense. In the sequel, for brevity, we say that a weak solution u is *strong* if

$$u \in L^s(0, T; L^q(\Omega)), \quad (2)$$

for some exponents s and q , $q \geq 3$, satisfying $\lambda(s, q) \leq 1$, where

$$\lambda(s, q) \equiv \frac{2}{s} + \frac{3}{q}. \quad (3)$$

In the following $\omega = \nabla \times u$ denotes the *vorticity* field. By a well known Sobolev's embedding theorem equation (2) and equation

$$\omega \in L^s(0, T; L^r(\Omega)) \quad (4)$$

are related by

$$\frac{1}{q} = \frac{1}{r} - \frac{1}{3}. \quad (5)$$

Consequently, solutions are strong if (4) holds for

$$\frac{2}{s} + \frac{3}{r} \leq 2. \quad (6)$$

In particular, solutions are strong if

$$\omega \in L^\infty(0, T; L^r(\Omega)) \quad (7)$$

for some $r \geq \frac{3}{2}$. This point has a main role in the sequel.

Let's go inside the core of this notes. We set

$$\theta(x, y, t) \stackrel{def}{=} \angle(\omega(x, t), \omega(y, t)),$$

where the symbol " \angle " denotes the amplitude of the angle between two vectors. We are interested in studying possible regularization effects of β -Hölder continuity assumptions on the direction of the vorticity, namely

$$\sin \theta(x, y, t) \leq c|x - y|^\beta \quad (8)$$

in $\mathbb{R}^3 \times \mathbb{R}^3 \times (0, T)$, for some $\beta \in (0, 1/2]$.

Our study concerns mainly the possible relations between the above assumption and regularity results like (7). It is well known, see [7], that assumption (8) with $\beta = \frac{1}{2}$ implies (7) with $r = 2$ (this situation is called here the *Hilbertian case*). Strong regularity follows. However, as already remarked, strong regularity is still guaranteed by (7), if merely $r \geq \frac{3}{2}$. This shows the weight of being able to prove (7) for some $r \geq \frac{3}{2}$. It is worth noting that this result looks quite natural. In fact, a strictly decreasing perturbation of β near $\frac{1}{2}$ should induce a corresponding strictly decreasing perturbation of r near 2. In particular, values $r \in (\frac{3}{2}, 2)$ would be obtained. Consequently strong regularity would follow for $\beta < \frac{1}{2}$. But, on the opposite side, this regularity result would be in some contrast with the opinion supported by us in a previous publication, see the appendix in reference [6], which predicts that strong regularity under the $\beta = \frac{1}{2}$ assumption should be "as strong as" the classical sufficient condition (2) for $\lambda = 1$. Roughly speaking, to improve the $\beta = \frac{1}{2}$ sufficient condition for strong regularity could be as hard as to extend the sufficient condition (2) to values $\lambda > 1$.

In the sequel we follow the perturbation strategy refereed above, namely, generating a strictly decreasing perturbation of β , in the proof of the Hilbertian case $(\beta, r) = (\frac{1}{2}, 2)$, and waiting for a corresponding drop of the value of the exponent r . This strategy requires to have in hands a suitable extension of the known $r = 2$ proof, which works for values $r \neq 2$. Suitable means here that the case $r = 2$ must be "perfectly" embedded in the new proof. Roughly speaking, with "continuity of argumentation" with respect

to the variable r . Actually the extension below does not require $r \neq 2$. The construction of this more general proof is a main point in the sequel.

It is worth noting that the smallest value β obtained by us below, which guarantees (7) regularity for exponents $r \in (1, 2]$, is still $\beta = \frac{1}{2}$. We believe that this "negative" conclusion is not due to an insufficiency of the above extended new proof, but is due to more subtle reasons. From one side, the conclusion reinforces our previous claim in favor of the real, not merely technical, significance of the particular value $\beta = \frac{1}{2}$. On the other side, it also shows a partial failure of a very natural attempt to improve the $\beta = \frac{1}{2}$ sufficient condition for regularity, by means of a perturbation approach. Summing up, a good understanding of the full phenomena persists. In particular, the link between β -Hölder continuity assumptions on the direction of the vorticity, for $\beta < \frac{1}{2}$, and regularity results of type $\omega \in L^s(L^r)$ is still an open problem. We hope that the specific approach presented below should help readers interested in carrying on this challenging investigation.

2 Some related known results.

We start by some references, strongly related to the present notes by methods of proof. We begin by recalling the very fundamental pioneering paper [16], by P.Constantin and Ch.Fefferman, where the authors prove, in particular, that solutions to the evolution Navier-Stokes equations in the whole space are smooth if the direction of the vorticity is Lipschitz continuous with respect to the space variables, namely assumption (8) for $\beta = 1$. This condition is assumed for almost all x and y in R^3 , and almost all t in $(0, T)$. Actually, in [16], the assumption is merely required for points x and y where the vorticity at both x and y is larger than a given, arbitrary constant k . This improvement was, or can be, extended in the same way to many subsequent papers on the subject. It is also easily show that assumption (8) can be restricted to couples of points x and y satisfying $|x - y| < \delta$, for an arbitrary positive constant δ .

In reference [7] L.C. Berselli and the present author showed, in particular, that regularity still holds in the whole space by replacing Lipschitz continuity by $\frac{1}{2}$ -Hölder continuity. This is, up to now, the strongest result in the literature. In the subsequent paper [3], by a trivial modification of the proof shown in [7], the following result is proved. Let $\beta \in (0, 1/2]$ be given. Further, assume that condition (8) holds and, in addition, that $\omega \in L^2(L^r)$ where $r = \frac{3}{\beta+1}$. Under these assumptions it was shown that the solution u is strong. In particular, the sufficiency of condition $\beta = \frac{1}{2}$ for regularity

is reobtained, since $\omega \in L^2(L^2)$ holds for any weak solution. This result is particularly related to the problem treated in the sequel.

Concerning other related papers, we start by recalling reference [4] where we extended the above kind of results to the half-space $\Omega = \mathbb{R}_+^3$, endowed with the slip boundary condition (“stress-free” boundary condition)

$$\begin{cases} u \cdot n = 0, \\ \omega \times n = 0, \end{cases} \quad (9)$$

where n is normal to the boundary. In reference [8], L.C. Berselli and the author succeed in extending this result to the case in which $\Omega \subset \mathbb{R}^3$ is an open, bounded set with a smooth boundary, by appealing to suitable representation formulas for Green’s matrices. In reference [9] regularity is proved by replacing continuity requirements on $\sin \theta(x, y, t)$ by a smallness assumption. Essentially, it is proved that there is a sufficiently small constant C_1 such that regularity holds if $\sin \theta(x, y, t) \leq C_1$. Clearly, there are many very interesting papers related to our contributions. We recall here, without any claim of completeness, references [5], [6], [10], [11], [12], [13], [14], [15], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26] [27], [28], [30].

3 The main new estimate.

In the following $f(s)$ denotes a real continuous differentiable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. We set

$$F(s) = \int_0^s f(\tau) d\tau.$$

Hence $F'(s) = f(s)$. By applying the curl operator to equation (1) we get the well-known equation

$$\omega_t + (u \cdot \nabla) \omega - \nu \Delta \omega = (\omega \cdot \nabla) u.$$

Scalar multiplication by $f(|\omega|^2) \omega$, integration in \mathbb{R}^3 , and integrations by parts easily show that

$$\frac{1}{2} \frac{d}{dt} \int F(|\omega|^2) dx - \int f(|\omega|^2) \Delta \omega \cdot \omega dx = \int f(|\omega|^2) (\omega \cdot \nabla) u \cdot \omega dx. \quad (10)$$

Non-labeled integrals are over \mathbb{R}^3 . Straightforward calculations show that

$$- \int f(|\omega|^2) \Delta \omega \cdot \omega dx = \int f(|\omega|^2) |\nabla \omega|^2 dx + 2 \int f'(|\omega|^2) |\omega|^2 |\nabla \omega|^2 dx.$$

Hence

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int F(|\omega|^2) dx + \int f(|\omega|^2) |\nabla \omega|^2 dx \\
& \leq 2 \int f'(|\omega|^2) |\omega|^2 |\nabla \omega|^2 dx + \int f(|\omega|^2) (\omega \cdot \nabla) u \cdot \omega dx.
\end{aligned} \tag{11}$$

In these notes we are interested in the particular case $f(s) = s^{-\alpha}$. In the Hilbertian case, in which $\alpha = 0$, many of the devices used in the sequel are superfluous.

It is useful to start by considering the approximation functions

$$f_\epsilon(s) = (\epsilon + s)^{-\alpha}, \tag{12}$$

where $\epsilon > 0$, and $0 \leq \alpha \leq \frac{1}{2}$. Note that $f'_\epsilon(s) < 0$. Straightforward calculations show that the absolute value of the first integral on the right hand side of equation (11) is bounded by α times the second integral on the left hand side of the same equation. So, one has

$$\begin{aligned}
& \frac{1}{2(1-\alpha)} \frac{d}{dt} \int (\epsilon + |\omega|^2)^{1-\alpha} dx + (1-2\alpha) \int (\epsilon + |\omega|^2)^{-\alpha} |\nabla \omega|^2 dx \\
& \leq \int (\epsilon + |\omega|^2)^{-\alpha} |\mathcal{K}(x)| dx
\end{aligned} \tag{13}$$

where

$$\mathcal{K}(x) := ((\omega \cdot \nabla) u \cdot \omega)(x). \tag{14}$$

Next we estimate from below the second integral on the left hand side of equation (13) (see [1] and [2] for similar manipulations). One has

$$(\epsilon + |\omega|^2)^{-\alpha} |\nabla |\omega||^2 = \frac{1}{(1-\alpha)^2} \frac{|\omega|^{2\alpha}}{(\epsilon + |\omega|^2)^\alpha} |\nabla(|\omega|^{1-\alpha})|^2. \tag{15}$$

Since $|\nabla \omega| \geq |\nabla |\omega||$, it follows from equation (13) that

$$\begin{aligned}
& \frac{1}{2(1-\alpha)} \frac{d}{dt} \int (\epsilon + |\omega|^2)^{1-\alpha} dx + \frac{(1-2\alpha)}{(1-\alpha)^2} \int |\nabla(|\omega|^{1-\alpha})|^2 dx \\
& \leq \int (\epsilon + |\omega|^2)^{-\alpha} |\mathcal{K}(x)| dx.
\end{aligned}$$

By letting $\epsilon \rightarrow 0$ one gets

$$\begin{aligned} & \frac{1}{2(1-\alpha)} \frac{d}{dt} \|\omega\|_{2(1-\alpha)}^{2(1-\alpha)} + \frac{(1-2\alpha)}{(1-\alpha)^2} \|\nabla(|\omega|^{1-\alpha})\|_2^2 \\ & \leq \int |\omega|^{-2\alpha} |\mathcal{K}(x)| dx \end{aligned} \quad (16)$$

which, for $\alpha = 0$, is precisely the estimate obtained in the Hilbertian case. Now we recall that, in the Hilbertian case, one appeals to the Sobolev's embedding $\|g\|_6 \leq c_0 \|\nabla g\|_2$ to allow substitution of $\|\nabla \omega\|_2$ by $\|\omega\|_6$. In the more general case considered here we appeal to the same device, by applying the above Sobolev's estimate to the function $g = |\omega|^{1-\alpha}$. After this device, equation (16) reads

$$\frac{d}{dt} \|\omega\|_{2(1-\alpha)}^{2(1-\alpha)} + c_1 \|\omega\|_{6(1-\alpha)}^{2(1-\alpha)} \leq \int |\omega|^{-2\alpha} |\mathcal{K}(x)| dx. \quad (17)$$

The symbol c , and similar, may denote distinct positive constants.

4 Estimating the nonlinear term by a Riesz potential.

In this section we estimate the left hand side of equation (16) by means of a related Riesz potential. This is one of the main ideas introduced by Constantin and Fefferman in [16], and took again in [7]. We follow here the presentation given in reference [6] (where bounded domains are considered). See also [8].

Since $-\Delta u = \nabla \times (\nabla \times u) - \nabla(\nabla \cdot u)$, it follows that

$$-\Delta u = \nabla \times \omega \quad \text{in } \mathbb{R}^3, \quad (18)$$

for each t . So

$$u(x) = \int G(x, y) (\nabla \times \omega)(y) dy, \quad (19)$$

where

$$G(x, y) = \frac{1}{4\pi |x - y|}.$$

In particular

$$\left| \frac{\partial^2 G(x, y)}{\partial y_k \partial x_i} \right| \leq \frac{c}{|x - y|^3}. \quad (20)$$

Set, for each triad (j, k, l) , $j, k, l \in \{1, 2, 3\}$,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation,} \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation,} \\ 0 & \text{if two indexes are equal.} \end{cases}$$

These are the components of the totally anti-symmetric Ricci tensor. One has

$$(a \times b)_j = \epsilon_{jkl} a_k b_l, \quad (\nabla \times v)_j = \epsilon_{jkl} \partial_k v_l. \quad (21)$$

The usual convention about summation of repeated indexes is assumed.

In particular

$$\left| \frac{\partial^2 G(x, y)}{\partial y_k \partial x_i} \right| \leq \frac{c}{|x - y|^3}. \quad (22)$$

By considering in equation (19) a single component u_j , and by appealing to (21), an integration by parts yields

$$u_j(x) = \int G(x, y) \epsilon_{jkl} \partial_k \omega_l(y) dy = - \int \epsilon_{jkl} \frac{\partial G(x, y)}{\partial y_k} \omega_l(y) dy.$$

Hence

$$\frac{\partial u_j(x)}{\partial x_i} = -P.V. \int \epsilon_{jkl} \frac{\partial^2 G(x, y)}{\partial x_i \partial y_k} \omega_l(y) dy.$$

It readily follows that

$$\mathcal{K}(x) = - \int \epsilon_{jkl} \frac{\partial^2 G(x, y)}{\partial y_k \partial x_i} \omega_i(x) \omega_j(x) \omega_l(y) dy.$$

Since $-\epsilon_{jkl} \omega_j(x) \omega_l(y) = (\omega_j(x) \times \omega_l(y))_k$, it follows that

$$\mathcal{K}(x) = P.V. \int \frac{\partial^2 G(x, y)}{\partial y_k \partial x_i} \omega_i(x) (\omega_j(x) \times \omega_l(y))_k dy.$$

By appealing to (22) one shows that

$$|\mathcal{K}(x)| \leq \int \frac{c}{|x - y|^3} |\omega(x)|^2 |\omega(y)| \sin \theta(x, y, t) dy. \quad (23)$$

Now we appeal to the main assumption (8), where $\beta \in [0, 1/2]$. By appealing to (23) one gets

$$|\mathcal{K}(x)| \leq c |\omega(x)|^2 I(x), \quad (24)$$

where

$$I(x) = \int_{\Omega} |\omega(y)| \frac{dy}{|x - y|^{3-\beta}}$$

is the Riesz potential in \mathbb{R}^3 . Recall that (see [29]) if $0 < \beta < 3$, and if $\omega \in L^{\widehat{r}}(\Omega)$ for some $1 < \widehat{r} < 3$, then

$$\|I\|_q \leq c \|\omega\|_{\widehat{r}}, \quad (25)$$

where

$$1/q = 1/\widehat{r} - \beta/3.$$

In particular, by (24), the right hand side of equation (17) satisfies the estimate

$$\int |\omega|^{-2\alpha} \mathcal{K}(x) dx \leq c \int |\omega|^r I(x) dx, \quad (26)$$

where $r = 2(1 - \alpha)$. So,

$$\frac{d}{dt} \|\omega\|_r^r + \|\omega\|_{3r}^r \leq c \int |\omega|^r I(x) dx, \quad (27)$$

for

$$1 < r \leq 2.$$

From now on we eliminate the above parameter α by appealing to the new exponent r . Next, by appealing to (25), we write

$$\frac{d}{dt} \|\omega\|_r^r + \|\omega\|_{3r}^r \leq c \|\omega\|_{\widehat{r}} \|\omega\|_{q'r}^r, \quad (28)$$

where

$$\frac{1}{q'} = 1 - \frac{1}{\widehat{r}} + \frac{\beta}{3}. \quad (29)$$

More precisely, by (16), we could write (not used in the sequel)

$$\frac{d}{dt} \|\omega\|_r^r + \|\nabla |\omega|^{\frac{r}{2}}\|_2^2 \leq c \|\omega\|_{\widehat{r}} \|\omega\|_{q'r}^r. \quad (30)$$

5 Conclusions.

Our final task, which is the main task in this section, is looking for pairs r and β such that the typical estimate

$$\|\omega\|_{\widehat{r}} \|\omega\|_{q'r}^r \leq C_\epsilon \|\omega\|_2^2 \|\omega\|_r^r + \epsilon \|\omega\|_{3r}^r \quad (31)$$

holds. As usual, the meaning of the above equation is that ϵ may be any positive, arbitrarily small real number, at the price of having a corresponding, arbitrarily large, value of C_ϵ . The motivation for this requirement is

standard. Assume that (31) holds for some pair of values r and β . Then, by appealing to (28), to (31), to

$$\|\omega(t)\|_2^2 \in L^1(0, T), \quad (32)$$

and to Gronwall's lemma, we show that

$$\omega \in L^\infty(0, T; L^r(\Omega)) \cap L^r(0, T; L^{3r}(\Omega)). \quad (33)$$

Even though $r \neq 2$, a central role in the right hand side of equation (31) is still required to the integrability exponent 2. The reason for this choice is that (32) is the strongest known estimate for the vorticity of weak solutions.

Our aim is now to find pairs $(r, \beta) \in (1, 2] \times (0, 1/2]$ such that (31) holds. It is convenient to start by recalling the way followed in the proofs in the Hilbertian case $r = 2$. In this case one had $(r, \beta, \hat{r}, q, q') = (2, 1/2, 2, 3, 3/2)$. Hence equation (28) reads

$$\frac{d}{dt} \|\omega\|_2^2 + \|\omega\|_6^2 \leq c \|\omega\|_2 \|\omega\|_3^2. \quad (34)$$

The next move in the classical proof was to appeal to the *interpolation inequality*

$$\|\omega\|_3^2 \leq \|\omega\|_2 \|\omega\|_6$$

which, together with (34), leads to the desired estimate

$$\frac{d}{dt} \|\omega\|_2^2 + \|\omega\|_6^2 \leq C_\epsilon \|\omega\|_2^2 \|\omega\|_2^2 + \epsilon \|\omega\|_6^2.$$

Following the same idea, we decompose both norms $\|\omega\|_{\hat{r}}$ and $\|\omega\|_{q'r}^r$ in the right hand side of (28), by appealing to interpolation. Note that, in the Hilbertian case, the choice $\hat{r} = 2$ in (31) looks obvious. On the contrary, in the more general situation treated here, a previous restriction could cut potential to the proofs. So we opted here for giving the largest possible width to the range of the parameter \hat{r} by assuming that

$$r \leq \hat{r} \leq 3r,$$

as suggested by the left hand side of (25). The same freedom will be given to $q'r$. More precisely, we start by considering parameters α, θ, γ and $\alpha', \theta', \gamma'$, in the interval $[0, 1]$, satisfying $\alpha + \theta + \gamma = \alpha' + \theta' + \gamma' = 1$, and related to the exponents $q'r$ and \hat{r} by the following equations:

$$\begin{cases} \frac{1}{q'r} = \frac{\alpha}{r} + \frac{\theta}{2} + \frac{\gamma}{3r}, \\ \frac{1}{\hat{r}} = \frac{\alpha'}{r} + \frac{\theta'}{2} + \frac{\gamma'}{3r}. \end{cases} \quad (35)$$

It follows, by interpolation, that

$$\begin{cases} \|\omega\|_{q'r} \leq \|\omega\|_r^\alpha \|\omega\|_2^\theta \|\omega\|_{3r}^\gamma, \\ \|\omega\|_{\hat{r}} \leq \|\omega\|_r^{\alpha'} \|\omega\|_2^{\theta'} \|\omega\|_{3r}^{\gamma'}. \end{cases} \quad (36)$$

The values of the above parameters will be fixed in the sequel. One has

$$B =: \|\omega\|_{q'r}^r \|\omega\|_{\hat{r}} \leq \|\omega\|_r^{\alpha' + \alpha r} \|\omega\|_2^{\theta' + \theta r} \|\omega\|_{3r}^{\gamma' + \gamma r}. \quad (37)$$

Next, by appealing to the dual exponents

$$\frac{r}{\gamma' + \gamma r}, \quad \frac{r}{(1 - \gamma)r - \gamma'},$$

we get

$$B \leq C_\epsilon \|\omega\|_r^{\frac{(\alpha' + \alpha r)r}{(1 - \gamma)r - \gamma'}} \|\omega\|_2^{\frac{(\theta' + \theta r)r}{(1 - \gamma)r - \gamma'}} + \epsilon \|\omega\|_{3r}^r. \quad (38)$$

We want

$$\frac{\alpha' + \alpha r}{(1 - \gamma)r - \gamma'} = 1, \quad \frac{\theta' + \theta r}{(1 - \gamma)r - \gamma'} = \frac{2}{r}, \quad (39)$$

since this leads to

$$\frac{d}{dt} \|\omega\|_r^r + \|\omega\|_{3r}^r \leq C_\epsilon \|\omega\|_2^2 \|\omega\|_r^r + \epsilon \|\omega\|_{3r}^r. \quad (40)$$

By setting $\gamma = 1 - (\alpha + \theta)$ and $\gamma' = 1 - (\alpha' + \theta')$ in the first equation (39), one easily shows that (39) is equivalent to

$$\begin{cases} \theta' + \theta r = 1, \\ \alpha' + \alpha r = \frac{r}{2}. \end{cases} \quad (41)$$

In addition, the exponents q' and \hat{r} must verify equation (29). This requirement is easily satisfied since the parameter β is still free. In other words, it gives the value of the Hölder exponent $\beta = \beta(r)$ that leads to the regularity result (33). Let's calculate this value. By appealing to the equation (29) and to the second equation (35), one shows that the first equation (35) can be written in the equivalent form

$$1 - \frac{\alpha'}{r} - \frac{\theta'}{2} - \frac{\gamma'}{3r} + \frac{\beta}{3} = \alpha + \frac{r}{2}\theta + \frac{\gamma}{3}.$$

Further, by replacing γ and γ' respectively by $1 - (\alpha + \theta)$ and $1 - (\alpha' + \theta')$, straightforward calculations lead to the desired expression of $\beta(r)$, namely

$$\beta(r) = \frac{2}{r}(\alpha' + \alpha r) + \left(\frac{3}{2} - \frac{1}{r}\right)(\theta' + \theta r) - 2 + \frac{1}{r}. \quad (42)$$

Lastly, by appealing to (41) and (42), we realize, in agreement to our prediction but also with some disappointment, that the exponent β obtained here does not depend on r . Actually, one gets

$$\beta = \frac{1}{2}. \quad (43)$$

Summing up, our attempt to feel out if the regularity result (33) may hold for $r < 2$ under a β -Hölder continuity assumption on the direction of the vorticity, for some $\beta < \frac{1}{2}$, has had here a negative reply. This conclusion supports the argument, defended in the appendix of [6], that values $\beta < \frac{1}{2}$ "does not imply strong regularity".

Note that (41) shows that "natural" pairs of values are $\alpha' = 0$, $\alpha = \frac{1}{2}$, and $\theta' = 1$, $\theta = 0$. Consequently, $\gamma = \frac{1}{2}$, and $\alpha' = \gamma' = 0$. These are exactly the values used in the proof of the Hilbertian case $r = 2$.

6 An open problem.

In these notes we have considered regularity results in the form shown by equation (7), hence with $s = \infty$. It could be that the desired additional regularity holds in the framework of finite values of s . A particular case would be to show that

$$\omega \in L^2(0, T; L^r(\Omega)),$$

for some $r > 2$. Weak solutions satisfy this requirement for $r = 2$. The following is a more general open problem:

Assume that a β -Hölder continuity assumption on the direction of the vorticity holds for some value $\beta < \frac{1}{2}$. To show that (2) holds for a couple of exponents s and q satisfying

$$\lambda(s, q) = \frac{2}{s} + \frac{3}{q} < \frac{3}{2}. \quad (44)$$

Note that $\lambda = 1$ means smoothness. The regularity required by equation (2), if $\lambda(s, q) = \frac{3}{2}$, corresponds to the maximum regularity known for generical weak solutions. Hence (44) would show some additional regularity to weak solutions, in terms of integrability, due to the above regularity assumption in terms of direction of vorticity. The possibility of this "transfer" of regularity looks quite natural, by taking into account that it holds in the Hilbertian case. Arriving to a reply to the above open problem, positive or negative, is a challenging open problem.

Equations (2) and (4) are related by (5). Roughly, the assumption (2) with exponents satisfying (44), is "near equivalent" to the assumption (4) with exponents satisfying

$$\frac{2}{s} + \frac{3}{r} < \frac{5}{2}. \quad (45)$$

However, it could be not negligible that (4) is stronger.

We may also consider reverse problems, like to verify whether regularity assumptions of type (2) imply, or not imply, regularity for the direction of the vorticity. Our guess is that there is not a "clean" reply to this problem.

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